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RESEARCH MEMORANDUM

LQ-CONTROL OF SAMPLED CONTINUOUS-TIME SYSTEMS

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Abstract

An algorithm in feedback form that generates piecewise constant controls which minimize a continuous-time quadratic cost functional with respect to a sampled continuous time-varying system is derived and computed. The system is linear and possesses an exogenous component. The cost functional is a quadratic tracking equation, involving a reference for both the output and control, and is considered over both a finite and infinite planning horizon. An economic example is included which demonstrates the application and numerical computation of the control algorithm in case of both a finite and infinite planning horizon.

1. Introduction

We study a problem that originates from the theory of economic stabilization. It concerns the design of a control policy yielding a prescribed behavior of the controlled system. Much research has been done on this problem, which is posed either as a continuous or a discrete-time control problem e.g. Tinbergen (1952), Aoki (1973), Pindyck (1973), Chow (1975), Turnowsky (1977), Maybeck (1982), Preston et. al. (1982) and Engwerda (1990-a).

From a practical point of view both the discrete-time and continuous-time problem formulation is ill-posed. The discrete-time formulation is ill-posed since usually an economic system evolves continuously in time. The continuous time problem is ill-posed since it generally assumes the control to vary continuously in time based on measurements which are performed continuously in time, which is an unrealistic assumption in case of economic systems. Measurements are only performed at certain time instances, therefore called sampling instances, and may be used to adjust the control which then remains unchanged until the next sampling instant. So the actual problem deals with the control of a sampled continuous-time system by means of piecewise constant controls. These control problems are generally referred to as sampled-data or digital control problems (Levis et al. 1971, De Koning 1980, Van Willigenburg 1990).

Levis et. al. (1971), Dorato and Levis (1971), De Koning (1980) and Maybeck (1982) demonstrated that the sampled-data regulator problem, i.e. a problem where the linear continuous-time system has no exogenous component and no references for the output and control variables appear in the quadratic cost functional, may be transformed into a so called *equivalent* discrete-time control problem. The equivalent discrete-time control problem is concerned with the minimization of the so called equivalent discrete-time cost functional subject to the so called equivalent discrete-time system. Both the equivalent discrete-time system and cost functional are obtained from the continuous-time system and cost functional through appropriate transformation. However this transformation is generally only partially used i.e. only the equivalent discrete-time system is used while a discrete-time cost functional *is searched for* which results in a satisfactory continuous-time behavior (Franklin and Powell, 1980). In case of the sampled-data regulator problem the equivalent discrete-time cost functional contains an extra (cross) term compared to the common discrete-time regulator cost functional. In other words the search for a discrete-time cost functional will never be successful! Obviously this search is a tedious, unnatural and unnecessary thing to perform if the original continuous-time control problem can be transformed into an *equivalent* discrete-time control problem. Besides the sampled-data regulator, the sampled-data tracker is known (Van Willigenburg 1990). In this case the cost functional includes a reference for the output variables only.

As pointed out by Engwerda (1990-a) the assumption that the system has no exogenous component while the cost functional does not include a reference for the control variables is rather restrictive given the objective to design economic control policies. This motivates the first part of this paper; the derivation and computation of the equivalent discrete-time control problem in case of linear-time varying systems with an exogenous component and quadratic cost functionals which may include references for both the output and control variables. A major requirement for a controller is to stabilize the system. Therefore the second part of this paper deals with the derivation and computation of the control algorithm in case the planning horizon is extended to infinity.

The paper is organized as follows. In section 2 we present the continuous-time optimal control problem and derive its discrete-time equivalent. In section 3 we present the solution of the equivalent discrete-time problem in case of a finite planning horizon and derive the solution in case the planning horizon is extended to infinity. In section 4 an economic example is presented which demonstrates the application and computation of the solution in case of both a finite and infinite planning horizon. In this section we also prove that whenever there exists an output path that can be ultimately tracked by some digital controller than our digital LQ-controller will attain this result too. We end the paper with remarks on straightforward generalizations that can be made on the presented theory in this paper.

2. Problem statement

Consider the finite dimensional continuous time-varying system

$$\mathring{y}(t) = A(t)y(t) + B(t)u(t) + C(t)x(t), \qquad t\in[t_0,t_N], \tag{1a}$$

where y is an n-dimensional output vector, u an m-dimensional control vector, x a p-dimensional uncontrollable deterministic vector, $A(t)$ is continuous and $B(t)$, $C(t)$, and $x(t)$ are piecewise continuous. Next we assume the continuous time-varying system is a sampled system i.e. we have the observations

$$y(t_k), \quad k=0,1,2,..,N-1, \tag{1b}$$

where t_k, $k=0,1,2,..,N-1$ are the, not necessarily equidistant, sampling instances. As clarified in the introduction we assume the control to be piecewise constant, i.e.

$$u(t) = u(t_k), \qquad t\in[t_k,t_{k+1}), \qquad k=0,1,2,..,N-1 \tag{1c}$$

The objective is to let $y(.)$ track an a priori determined reference trajectory $y^*(.)$ by choosing $u(.)$ in a suitable manner. We assume the initial values of the system $y(0)$ and the trajectory $x(.)$ to be known before $u(.)$ is chosen. To formalize our idea of tracking we introduce the quadratic cost functional

$$J(u(.),t_0,t_N) = (y(t_N)-y^*(t_N))^T H(y(t_N)-y^*(t_N)) + \int_{t_0}^{t_N} \{(y(t)-y^*(t))^T Q(t)(y(t)-y^*(t)) + (u(t)-u^*(t))^T R(t)(u(t)-u^*(t))\}\, dt \tag{1d}$$

where we assume $Q(.)$ and H to be symmetric semi-positive definite weighting matrices and $R(.)$ to be positive definite. Later on we will demonstrate that the positive definiteness of $R(.)$ may sometimes be relaxed to $R(.)$ being semi-positive definite. By minimizing this cost functional with respect to system *(1a,b)* we express the aim to track the prescribed output reference $y^*(.)$ using a control $u(.)$ which does not differ to much from a prescribed control reference (policy) $u^*(t)$. The choice of the weighting matrices reflects the relative importance of tracking the output and control references. Although not strictly necessary, it seems reasonable to assume $u^*(.)$ like the control $u(.)$ is piecewise constant.

It is well known (see e.g. Levis et. al. 1971) that the solution of the system *(1a,c)* is given by

$$y(t)=\phi(t,t_k)y(t_k)+\Gamma(t,t_k)u(t_k)+d(t,t_k),$$

$$t\in[t_k,t_{k+1}),\ k=0,1,..,N-1, \qquad (2)$$

where the transition matrix $\phi(t,t_k)$ is the solution of the matrix differential equation

$$d/dt\ \phi(t,t_k) = A(t)\ \phi(t,t_k),\ t\in[t_k,t_{k+1}),\ k=0,1,..,N-1, \qquad (3a)$$

with the initial condition

$$\phi(t_k,t_k) = I, \qquad (3b)$$

where I is the identity matrix. Furthermore

$$\Gamma(t,t_k) = \int_{t_k}^{t} \phi(t,s)B(s)ds, \qquad (4)$$

and finally

$$d(t,t_k) = \int_{t_k}^{t} \phi(t,s)C(s)x(s)ds. \qquad (5)$$

With $t=t_{k+1}$ in *(2)* we have

$$y_{k+1} = \phi_k y_k + \Gamma_k u_k + d_k, \qquad k=0,1,..,N-1, \qquad (6a)$$

where

$$y_k = y(t_k), \qquad (6b)$$

$$u_k = u(t_k), \tag{6c}$$

$$\phi_k = \phi(t_{k+1}, t_k), \tag{6d}$$

$$\Gamma_k = \Gamma(t_{k+1}, t_k), \tag{6e}$$

$$d_k = d(t_{k+1}, t_k). \tag{6f}$$

System *(6)* is called the equivalent discrete-time system of *(1a,b)* since the behavior of both systems coincides at the sampling instances. The cost functional *(1c)* equals

$$J(u(.),t_0,t_N) = (y(t_N)-y^*(t_N))^T H(y(t_N)-y^*(t_N)) + \sum_{k=0}^{N-1} \left[\int_{t_k}^{t_{k+1}} \{(y(t)-y^*(t))^T Q(t)(y(t)-y^*(t)) + (u(t)-u^*(t))^T R(t)(u(t)-u^*(t))\}\, dt \right]. \tag{7}$$

Using *(2)* and *(6)* equation *(7)* becomes

$$J(u(.),0,N) = (y_N-y^*(t_N))^T H(y_N-y^*(t_N)) + \sum_{k=0}^{N-1} \left\{ y_k^T Q_k y_k + 2y_k^T M_k u_k + u_k^T R_k u_k + 2y_k^T v_k + 2w_k^T u_k + z_k \right\}, \tag{8a}$$

where

$$Q_k = \int_{t_k}^{t_{k+1}} \phi^T(t,t_k)\, Q(t)\, \phi(t,t_k)\, dt, \tag{8b}$$

$$M_k = \int_{t_k}^{t_{k+1}} \phi^T(t,t_k)\, Q(t)\, \Gamma(t,t_k)\, dt, \tag{8c}$$

$$R_k = \int_{t_k}^{t_{k+1}} [\, \Gamma^T(t,t_k)\, Q(t)\, \Gamma(t,t_k) + R(t)\,]\, dt, \tag{8d}$$

$$v_k = \int_{t_k}^{t_{k+1}} \phi^T(t,t_k)\, Q(t)\, (d(t,t_k) - y^*(t))\, dt, \tag{8e}$$

$$w_k = \int_{t_k}^{t_{k+1}} [\, \Gamma^T(t,t_k)\, Q(t)\, (d(t,t_k) - y^*(t)) - R(t)\, u^*(t)\,]\, dt, \tag{8f}$$

$$z_k = \int_{t_k}^{t_{k+1}} (y^*(t) - d(t,t_k))^T Q(t)(y^*(t) - d(t,t_k)) + u^{*T}(t)R(t)u^*(t)\, dt. \tag{8g}$$

Starting from the sampled-data optimal control problem *(1)* we have arrived at the equivalent discrete-time version of the problem given by the equivalent discrete-time cost functional *(8)* and the equivalent discrete-time system *(6)*. Note that the equivalent discrete-time cost functional contains a cross term which is usually not included in discrete-time linear quadratic optimal control problems. They will therefore never result in an optimal continuous-time behavior!

3. Problem solution

In this section we present the solution of the equivalent discrete-time problem *(6)*, *(8)* derived from the sampled-data optimal control problem *(1)* in case N, the planning horizon of the problem, is finite. Finally we consider the minimization of *(8)* when N is extended to infinity, i.e. $\lim_{N\to\infty} J(u(.),0,N)$ subject to system *(6)*.

Theorem 1

The control sequence minimizing *(8)* with respect to *(6)* is given by

$$u_k = -G_{k,N}y_k - g_{k,N}, \tag{9a}$$

where

$$G_{k,N} = (R_k+\Gamma_k^T K_{k+1,N}\Gamma_k)^{-1}(\Gamma_k^T K_{k+1,N}\phi_k+M_k^T), \tag{9b}$$

$$g_{k,N} = (R_k+\Gamma_k^T K_{k+1,N}\Gamma_k)^{-1}(\Gamma_k^T K_{k+1,N}d_k-\Gamma_k^T h_{k+1,N}+w_k), \tag{9c}$$

while $K_{k,N}$ and $h_{k,N}$ are given by the following recursive equations

$$K_{k,N} = Q_k + \phi_k^T K_{k+1,N}\phi_k -$$

$$(\phi_k^T K_{k+1,N}\Gamma_k+M_k)(R_k+\Gamma_k^T K_{k+1,N}\Gamma_k)^{-1}(\Gamma_k^T K_{k+1,N}\phi_k+M_k^T),$$

$$K_{N,N} = H, \tag{9d}$$

$$h_{k,N} = (\phi_k-\Gamma_k G_{k,N})^T(h_{k+1,N}-K_{k+1,N}d_k) + G_{k,N}^T w_k - v_k,$$

$$h_{N,N} = Hy^*(t_N). \tag{9e}$$

Finally we have for the minimum cost over the interval [k,N]

$$\min J(u(.),k,N) = y_k^T G_{k,N}y_k-2y_k^T h_{k,N}+\alpha_{k,N}, \tag{9f}$$

where $\alpha_{k,N}$ is given by the recursion

$$\alpha_{k,N} = (K_{k+1,N}d_k - h_{k+1,N})^T (R_k + \Gamma_k^T K_{k+1,N}\Gamma_k)^{-1}\Gamma_k^* $$
$$(\Gamma_k^T(h_{k+1,N} - K_{k+1,N}d_k) - 2w_k) - 2d_k^T h_{k+1,N} + d_k^T K_{k+1,N} d_k -$$
$$w_k(R_k + \Gamma_k^T K_{k+1,N}\Gamma_k)^{-1} w_k^T + z_k + \alpha_{k+1,N},$$

$$\alpha_{N,N} = y^{*^T}(t_N) H y^*(t_N) \tag{9g}$$

□

Proof

One way to demonstrate the correctness of this theorem is to use the result of Engwerda (1990-a), theorem 1 with $u_k^* = -R_k^{-1}w_k$, $y_k^* = -Q_k^{-1}v_k$, $k=0,1,\dots,N-1$, $Q_N = H$, $G_k = I$, and Maybeck (1982, pp.73-76). Van Willigenburg (1991, pp.68-73), used dynamic programming to solve the same equivalent discrete-time problem with $x_k = y_k$, $L_k = -v_k^T$, $T_k = -w_k^T$, $X_k = z_k$, except for the fact that the system has no exogenous component but is corrupted by additive white noise. The exogenous component requires slight modification of the proof. After modification we also obtain *(9f)*, *(9g)*, an expression for the minimum costs explicit in the system and cost functional matrices. Moreover this proof does not require the invertibility of Q_k. We only need $Q_k \geq 0$, and $R_k > 0$. From *(8b)* observe that $Q(.) \geq 0$ implies $Q_k \geq 0$. Given $Q(.) \geq 0$ from *(8d)* observe that $R(.) > 0$ implies $R_k > 0$. However $R(.) \geq 0$ may also imply $R_k > 0$ if $\Gamma^T(t,t_k)R(t)\Gamma(t,t_k) > 0$ over some time-interval within $[t_k, t_{k+1})$.

□

We now concentrate on the minimization of $\lim_{N\to\infty} J(u(.),0,N)$ with respect to system *(6)*. From Kwakernaak et al. (1972) (see also Anderson et al. 1981 and Engwerda 1990-b) we have

Lemma 2

Let ϕ_k, Γ_k be uniformly completely controllable and $(\sqrt{Q_k}, \phi_k)$ uniformly completely reconstructable. Moreover assume $\phi_k, \Gamma_k, Q_k, M_k, R_k$ are bounded and $R_k \geq \beta I$ for some $\beta > 0$ and for all $k \geq 0$. Then $\lim_{N\to\infty} K_{k,N} = K_k$ and consequently $\lim_{N\to\infty} G_{k,N} = G_k$ exist for all $k \geq 0$ and the closed loop system

$$z_{k+1} = (\phi_k - \Gamma_k G_k) z_k \tag{10}$$

is exponentially stable. □

In the following we will call
$\alpha = \inf\{\lambda \mid \exists\gamma > 0$ such that $\|z_k\|_E \leq \gamma\lambda^k \|z_0\|_E$ for all $k \in \mathbb{N}$ and $z_0 \in \mathbb{R}^n\}$ the decay rate of the closed loop system *(10)*.

Analogous to corollary 2 in Engwerda (1990-a) we have

Theorem 3

Let all assumptions of lemma 2 be satisfied and d_k, v_k and w_k be such that for all $k \geq 0$,

$$h_k = \lim_{N\to\infty} h_{k,N} \tag{11}$$

exists. Then the optimal control minimizing $\lim_{N\to\infty} J(u(.),0,N)$ with respect to system (6) is given by

$$u_k = -(R_k + \Gamma_k^T K_{k+1} \Gamma_k)^{-1} (\Gamma_k^T K_{k+1} \phi_k + M_k^T) y_k$$

$$-(R_k + \Gamma_k^T K_{k+1} \Gamma_k)^{-1} (\Gamma_k^T K_{k+1} d_k - \Gamma_k^T h_{k+1} + w_k), \tag{12}$$

□

Theorem 4

Let all assumptions of lemma 2 be satisfied and assume the growth

rate of the variables d_k, v_k and w_k is smaller than $\beta<1/\alpha$, i.e., $||d_{k+1}||\leq\beta||d_k||$, $||v_{k+1}||\leq\beta||v_k||$ and $||w_{k+1}||\leq\beta||w_k||$.
Then $h_k = \lim_{N\to\infty} h_{k,N}$ exists and is given by

$$h_k = \sum_{l=k}^{\infty} \left\{(\phi-\Gamma G)^T(i,k)\right\} \left\{-(\phi_l-\Gamma_l G_l)^T K_{l+1} d_l + G_l^T w_l - v_l\right\} \tag{14}$$

where

$$(\phi-\Gamma G)(i,k) = (\phi_{l-1}-\Gamma_{l-1}G_{l-1})*(\phi_l-\Gamma_l G_l)*\ldots*(\phi_k-\Gamma_k G_k) \tag{15}$$

□

Summarizing, the solution of the equivalent discrete-time problem *(6)*, *(8)* is given by theorem 3, while theorem 4 states sufficient conditions for the solution to exist and determines h_k which is part of the solution.

Based on slight modification of the results of Van Willigenburg (1991) we are able to numerically compute the solution *(9)* to the general LQ problem *(1)* if the planning horizon is finite. If the horizon is extended to infinity from *(9d)*, *(9e)* observe that in general we have to perform an infinite number of computations which need an infinite number of data concerning the system and cost functional. However if all the conditions of theorem 4 are satisfied taking a sufficiently large horizon allows us to approximate the optimal control arbitrarily close. Loosely speaking if the conditions of theorem 4 are satisfied the outcome of the backward recursions *(9d)*, *(9e)* i.e. $h_{k,N}$ and $K_{k,N}$ are hardly influenced by $h_{k',N}$ and $K_{k',N}$ where $k'-k$ is a large positive integer, i.e. the outcome of the backward recursions is hardly influenced by far distant values. In that case taking a sufficiently large horizon and arbitrary initial values for the recursions *(9d)*, *(9e)* the outcome of *(9)* will approximate the solution arbitrarily close. This is demonstrated in the next section. A more detailed analysis of this phenomenon can be found in Engwerda (1992).

4. An economic example; the multiplier-accelerator model

The example presented in this section has the special property that it can be formulated both as a time-varying LQ problem (i.e. the system and/or cost functional matrices are time-varying) where the system possesses an exogenous component and as a standard time-invariant LQ regulator problem (i.e. the system and cost functional matrices are time-invariant and the system possesses no exogenous component). The solution to the latter problem in case of both continuous and sampled data, when the sampling interval is constant, can be computed using standard tools available from the Matlab Control Toolbox. This offers an alternative way to numerically compute the solution and therefore offers a possibility to numerically check our results.

Turnovsky (1972) showed how the optimal government expenditure in LQ sense is determined for the standard multiplier-accelerator model developed by Philips (1954). This model is described by the following three equations

$$Y = \gamma\, C + I + G_e + D, \tag{16a}$$
$$I = \alpha\, \mathring{C} - i\, \mathring{I}, \tag{16b}$$
$$\mathring{C} = \delta\, (Y - C), \tag{16c}$$

where Y denotes aggregate demand, C consumption, I investment, G_e government expenditure, D autonomous expenditure and the parameters γ, α, i and δ are assumed to be positive constants with $0 \le \gamma \le 1$. Assuming the government planner controls government expenditure, i.e. assuming G_e to be the control variable, the objective is to minimize the welfare function

$$J(G_e(.),t_0,t_N) = [f_1(C(t_N)-C^*(t_n))^2+f_2(G_e(t_N)-G_e^*(t_n))^2] +$$

$$\int_{t_0}^{t_N} [m_1(C-C^*)^2 + m_2(G_e-G_e^*)^2 + n(\mathring{G}_e)^2]dt, \tag{17}$$

where f_1, f_2, m_1, m_2 and n are positive constants while C^* and G_e^* are prescribed constant target values for consumption and government expenditure respectively. As a special case we examine the optimal control policy for the instantaneous accelerator model (Turnovsky, 1972) i.e. we take $i=0$ and consider the time-discounted version of the welfare function *(17)* with $f_1=f_2=n=0$ and $t_N=\infty$. Introducing,

$$x=C-C^*,\ g=G_e-G_e^*,\ \beta=-\delta/(1-\alpha\delta),\ \sigma=1-\delta,\ ex=\beta(\sigma C^*-G_e^*-D), \tag{18}$$

the LQ tracking problem *(16)*, *(17)* is transformed into

$$\mathring{x} = \beta\sigma x - \beta g + ex, \tag{19}$$

$$J(G_e(.),0,\infty) = \int_0^\infty e^{-\rho t}(m_1 x^2+m_2 g^2)dt, \tag{20}$$

which constitutes a time-varying LQ regulator problem where the system *(19)* possesses an exogenous component. Since D is constant from *(18)* observe that the variable ex will be constant too and that in general simultaneous tracking of both target consumption and government expenditure is impossible (compare e.g. with Engwerda, 1991). Introducing

$$y = (y_1,\ y_2)^T, \tag{21a}$$

$$y_1 = e^{-0.5\rho t}x, \tag{21b}$$

$$y_2 = e^{-0.5\rho t}ex, \tag{21c}$$

$$u = e^{-0.5\rho t}g, \tag{21d}$$

from equation *(19)* we obtain

$$\mathring{y} = \begin{bmatrix} \beta\sigma-0.5\rho & 1 \\ 0 & -0.5\rho \end{bmatrix} y + \begin{bmatrix} -\beta \\ 0 \end{bmatrix} u, \tag{22}$$

and from *(20)*

$$J = \int_0^\infty y^T \begin{bmatrix} m_1 & 0 \\ 0 & 0 \end{bmatrix} y + m_2 u^2 \, dt. \tag{23}$$

From *(22)* and *(23)* observe that the time-varying LQ regulator problem *(19)*,*(20)* can be reformulated as a standard time-invariant LQ regulator problem *(22)*, *(23)* (i.e. with time-invariant system and cost functional matrices and where the system does not possess an exogenous component).

We will first assume the state y to be continuously available. Application of the well known solution to the standard continuous regulator problem (see e.g. Lewis 1986) to *(22)*, *(23)* in this case results in the following feedback law

$$g = \beta/m_2(p_{11}x + p_{12}ex), \tag{24a}$$

where p_{11} is the positive solution to the quadratic algebraic Riccati equation

$$\beta^2 p_{11}^2/m_2 - (2\beta\sigma - \rho)p_{11} - m_1 = 0, \tag{24b}$$

and

$$p_{12} = -p_{11}/(\beta\sigma - \rho - p_{11}\beta^2/m_2). \tag{24c}$$

Substitution of the optimal control policy *(24)* into *(19)* yields the closed loop error equation for consumption;

$$\Delta\mathring{x} = (\beta\sigma - p_{11}\beta^2/m_2)\ \Delta x + (\beta\sigma - \rho)\ ex/(\beta\sigma - \rho - p_{11}\beta^2/m_2),$$

(25)

from which the steady state error in consumption is computed (note that $\beta\sigma - p_{11}\beta^2/m_2 < 0$)

$$\Delta\bar{x} = (\beta\sigma - \rho)\, ex/((\beta\sigma - \rho - p_{11}\beta^2/m_2)(-\beta\sigma + p_{11}\beta^2/m_2)). \tag{26a}$$

The corresponding steady state error in government expenditure in this case is given by,

$$\Delta\bar{g} = \beta/m_2(p_{11}\bar{x} + p_{12}ex), \tag{26b}$$

The solution *(24)* of the continuous regulator problem *(22)*, *(23)* with time-invariant system and cost functional matrices may be numerically computed using the function lqr.m from the Control Toolbox of Matlab. A simulation of the controlled system over the first 15 years is shown in figure 1 where we used the following parameter values

$$\gamma = 0.9,\ \alpha = 0.9,\ \delta = 0.4,\ m_1 = 2.0,\ m_2 = 1.0,\ \rho = 0.1. \tag{27a}$$

From *(27a)* and *(18)* we obtain

$$\beta = -0.06250,\ \sigma = 0.1,\ ex = -25.0. \tag{27b}$$

The target values for consumption and government expenditure where chosen to be,

$$C^* = 400,\ G^* = 100. \tag{27c}$$

Finally we chose for the initial value of consumption

$$C(0) = 380. \tag{27d}$$

The simulation results may be verified against the formulas *(24)* and *(26)*.

Next we will assume the economy to be quarterly sampled i.e. the state y to be available only at equidistant time instances t_k, $k=0,1,2,\dots$ where

$$t_{k+1}-t_k = 0.25,\ k=0,1,2,\dots\dots \tag{28}$$

If we consider the formulation *(22), (23)*, i.e. the formulation as a standard time-invariant regulator problem since the sampling interval is constant and we have an infinite planning horizon we may use the function lqrd.m from the Control Toolbox of Matlab to numerically compute the solution. A simulation of the resulting control system over the first 15 years is shown in figure 2.

Next we numerically compute the solution based on the time-varying problem formulation *(19),(20)* taking a "large" horizon, in this case 30 years and taking zero as the initial value for both the recursions *(9d), (9e)*, using the modified results of van Willigenburg (1991). A simulation of the resulting control system is shown in figure 3. We compared the solution to the previous one over the first 15 years. The differences between the two numerical solutions are shown in figure 4. The optimal control is approximated within 0.7%, the optimal state trajectory within 0.2%. This demonstrates that the choice of a sufficiently large horizon results in an arbitrary close approximation of the solution of the LQ problem *(1)* in case of an infinite planning horizon if the conditions of theorem 4 are satisfied.

From the simulation of the sampled and continuous-time control system we observe that the errors for consumption and government expenditure show an equal steady state behavior when time goes to infinity. This is caused by the fact that the problem can be expressed as a standard time-invariant LQ regulator problem with an infinite planning horizon. Obviously in general these errors will defer because the control in case of the sampled system is constrained to be piecewise constant. Another particular case in which these errors coïncide, and in fact are equal to zero, when time goes to infinity, occurs if *ex* is equal to zero. This particular case is generalized in the next theorem. Before we state this theorem we have to introduce the notion of smooth tracking.

Definition 5

$y^*(.)$ smoothly tracks $y(.)$ if both $e(t)=y^*(t)-y(t)$ and its derivative converge to zero when time goes to infinity.

□

Lemma 6

Let $y^*(.)$ and $u^*(.)$ be prescribed reference trajectories for output and control variables respectively and assume that $y^*(.)$ is differentiable. Then in model *(1a)* there exists a control sequence $u(.)$ which converges to $u^*(.)$ such that the output is smoothly tracked iff $y^*(.)$ and $u^*(.)$ satisfy the following relationship

$$\mathring{y}^*(t) = A(t)y^*(t) + B(t)u^*(t) + C(t)x(t) + \varepsilon(t) \qquad (29)$$

where $\varepsilon(t)\to 0$ when $t\to\infty$.

□

The proof of this lemma follows directly from the observation that

$$\overset{\circ}{(y(t)-y^*(t))} = A(t)(y(t)-y^*(t)) + B(t)(u(t)-u^*(t))$$

$$+ A(t)y^*(t) + B(t)u^*(t) + C(t)x(t) - \mathring{y}^*(t) \qquad (30)$$

Definition 7

All output trajectories of the form *(29)* with $\varepsilon(t)\to 0$ when $t\to\infty$, where $u^*(.)$ is an admissible control function, are called asymptotically admissible output trajectories. Any admissible control function $u(.)$ that succeeds in smoothly tracking such an output trajectory is called a successful controller.

□

Note that in definition 7 we speak of admissible control functions, since in case of the sampled system *(1)* admissible controls are of the form *(1c)*. This implies that all asymptotically admissible output trajectories of the sampled system *(1)* are a subset of all asymptotically admissible output trajectories of the continuous time system *(1a)* which lacks the control constraint *(1c)*.

Theorem 8

Let all assumptions of lemma 2 be satisfied. Then the sampled LQ tracker *(12)* is a successful controller for any asymptotically admissible output trajectory of the sampled system *(1)*.

Proof

Define the output error and control error as $e(t)=y(t)-y^*(t)$ and $\Delta u(t)=u(t)-u^*(t)$ respectively. Then the minimization of $\lim_{N\to\infty} J(u(.),o,N)$ subject to the system *(1)* can be rewritten as the minimization of

$$\lim_{N\to\infty} J(u(.),o,N) =$$

$$\lim_{N\to\infty} \left\{ e_N H e_N + \sum_{k=0}^{N-1} \left\{ e_k^T Q_k e_k + 2e_k^T M_k \Delta u_k + \Delta u_k^T R_k \Delta u_k \right.\right.$$

$$\left.\left. + 2e_k^T \bar{v}_k + 2\bar{w}_k^T \Delta u_k + \bar{z}_k \right\} \right\}, \qquad (31a)$$

where

$$\bar{v}_k = \int_{t_k}^{t_{k+1}} \phi^T(t,t_k)\, Q(t)\, \bar{d}(t,t_k), \qquad (31b)$$

$$\bar{w}_k = \int_{t_k}^{t_{k+1}} \Gamma^T(t,t_k)\; Q(t)\; \bar{d}(t,t_k), \tag{31b}$$

$$\bar{z}_k = \int_{t_k}^{t_{k+1}} \bar{d}^T(t,t_k)\; Q(t)\; \bar{d}(t,t_k), \tag{31b}$$

$$\bar{d}(t,t_k) = \int_{t_k}^{t} \phi^T(t,s)\; \varepsilon(s)\; ds, \tag{31c}$$

with respect to the system

$$e_{k+1} = \phi_k e_k + \Gamma_k \Delta u_k + \bar{d}_k. \tag{32}$$

According to theorem 3 and 4 the optimal control is given by

$$\begin{aligned}\Delta u_k = &-(R_k+\Gamma_k^T K_{k+1}\Gamma_k)^{-1}(\Gamma_k^T K_{k+1}\phi_k+M_k^T)e_k \\ &-(R_k+\Gamma_k^T K_{k+1}\Gamma_k)^{-1}(\Gamma_k^T K_{k+1}\bar{d}_k-\Gamma_k^T\bar{h}_{k+1}+\bar{w}_k),\end{aligned} \tag{33a}$$

with

$$\bar{h}_k = \sum_{i=k}^{\infty}\left\{(\phi-\Gamma G)^T(i,k)\right\}\left\{-(\phi_i-\Gamma_i G_i)^T K_{i+1} d_i + G_i^T w_i - v_i\right\}. \tag{33b}$$

The closed loop system therefore reads

$$\begin{aligned}e_{k+1} = &(\phi_k-\Gamma_k(R_k+\Gamma_k^T K_{k+1}\Gamma_k)^{-1}(\Gamma_k^T K_{k+1}\phi_k+M_k^T))e_k - \\ &\Gamma_k(R_k+\Gamma_k^T K_{k+1}\Gamma_k)^{-1}(\Gamma_k^T K_{k+1}\bar{d}_k-\Gamma_k^T\bar{h}_{k+1}+\bar{w}_k)\end{aligned} \tag{34}$$

Since $\bar{d}_k$, $\bar{v}_k$ and $\bar{w}_k$ converge to zero for $k\to\infty$, $\bar{h}_k$ converges to zero too. Consequently the closed loop system matrix is stable! e_k

converges to zero together with Δu_k when $k \to \infty$ which completes the proof.

□

5. Aftersight

We derived and numerically computed the solution to an LQ control problem for a sampled continuous time-varying system with an exogenous component and piecewise constant controls, given a continuous-time quadratic cost functional with references for both the output and control, in case of both a finite and infinite planning horizon. The solution was obtained in two steps. The sampled continuous-time control problem, where the control is constrained to be piecewise constant, was transformed into an *unconstrained equivalent* discrete-time control problem. By combining and extending results from Maybeck (1982), Engwerda (1990-a) and Van Willigenburg (1991) we showed that given some conditions on the equivalent discrete-time system and cost functional an optimal control algorithm exists and may be numerically computed in case of a finite planning horizon.

The motivation to study the case of an infinite planning horizon is to obtain a stabilizing controller. In case of an infinite planning horizon additional conditions were presented that guarantee the existence of a solution together with the stability of the closed loop system. These conditions also guarantee the convergence of a solution with a sufficiently large planning horizon to the solution of the problem with an infinite planning horizon. Therefore the numerical computation of the solution in case of an infinite planning horizon may be realized by computing the solution to a problem with a sufficiently large planning horizon, which we demonstrated through an economic example.

We have deliberately kept the performed analysis easy. The conditions of uniform complete controllability and uniform complete reconstructability may be relaxed (See e.g. Engwerda, 1990-b). Slight modification of Van Willigenburg (1991) shows the results can easily be extended to a situation where the system is corrupted by additive white noise, while the output information is incomplete and corrupted by additive white noise. Finally a number of conditions may be relaxed if the system is time-invariant.

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Figure 1a:Solution in case of continuous data

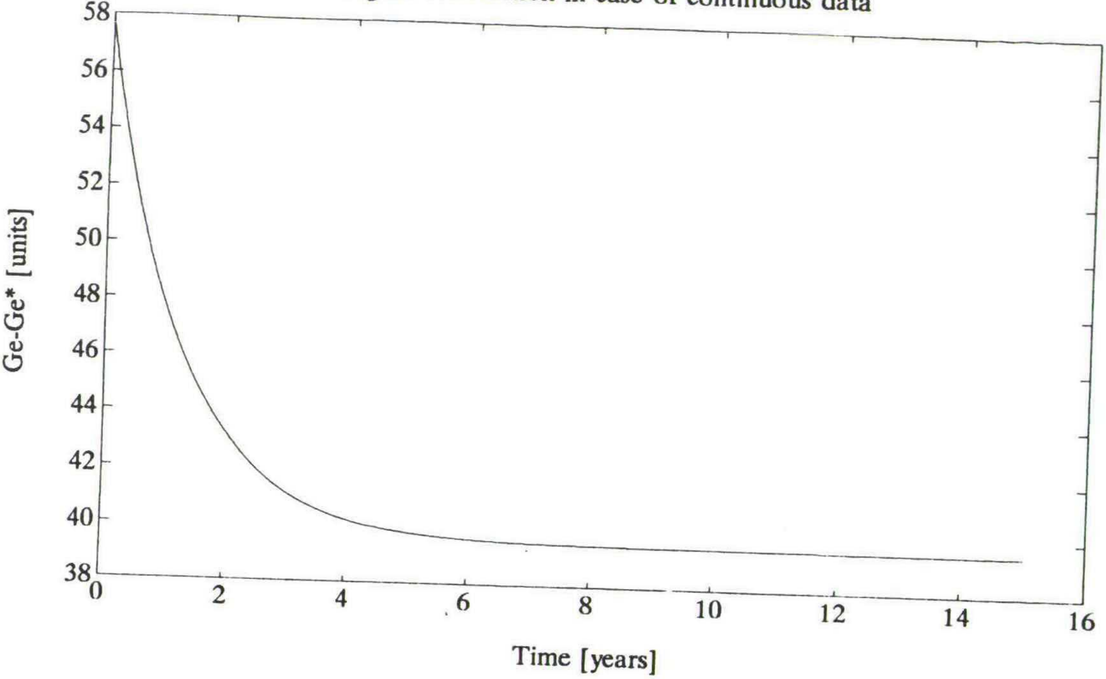


Figure 1b:Solution in case of continuous data

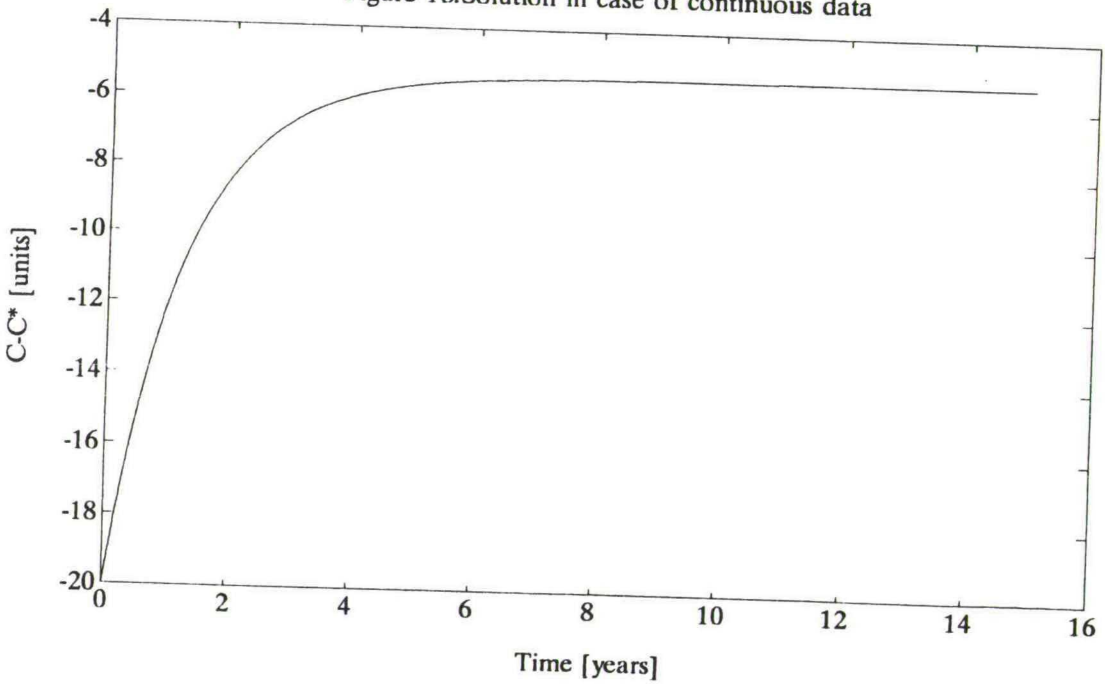


Figure 2a: Numerical solution of time-invariant problem formulation

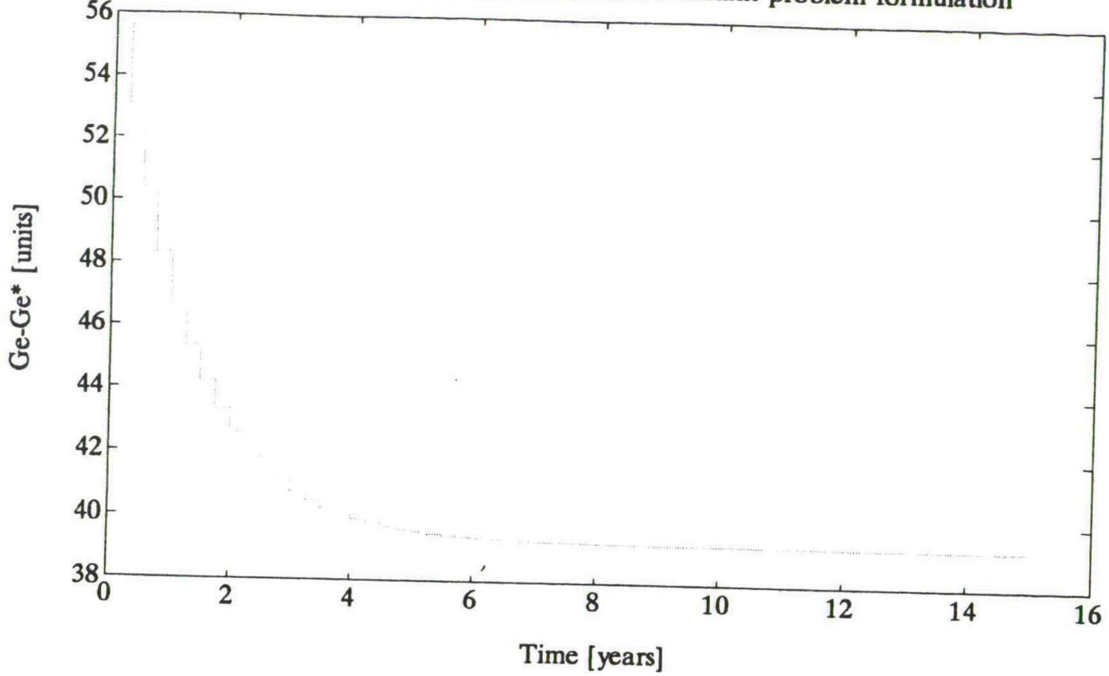


Figure 2b:Numerical solutions of time-invariant problem formulation

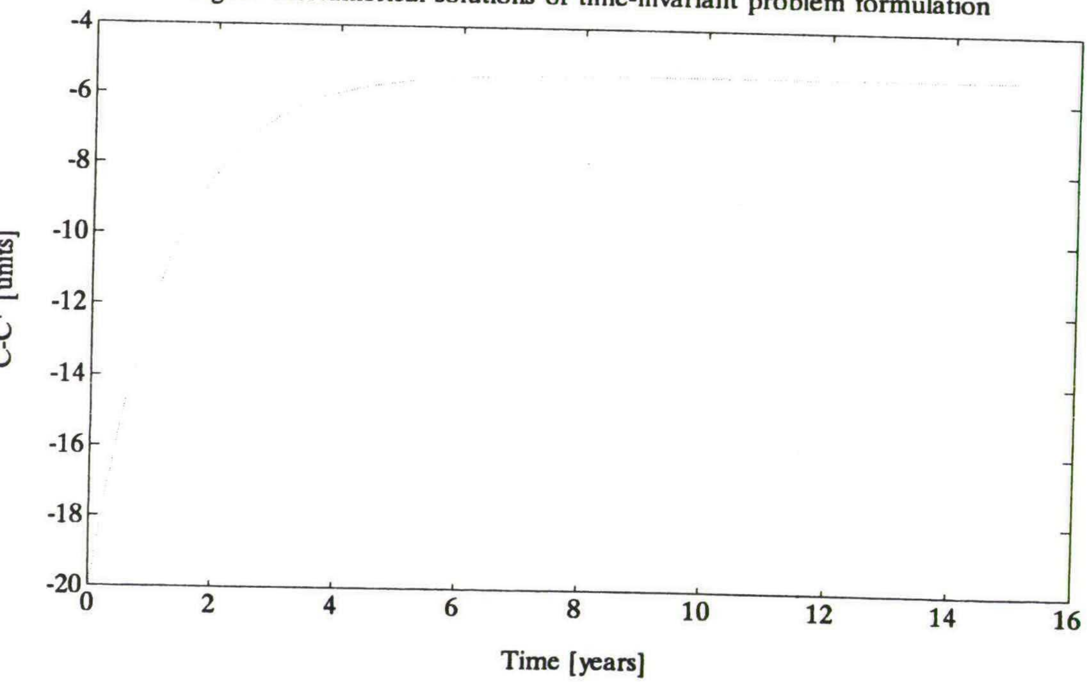


Figure 3a: Numerical solution of time-varying problem formulation

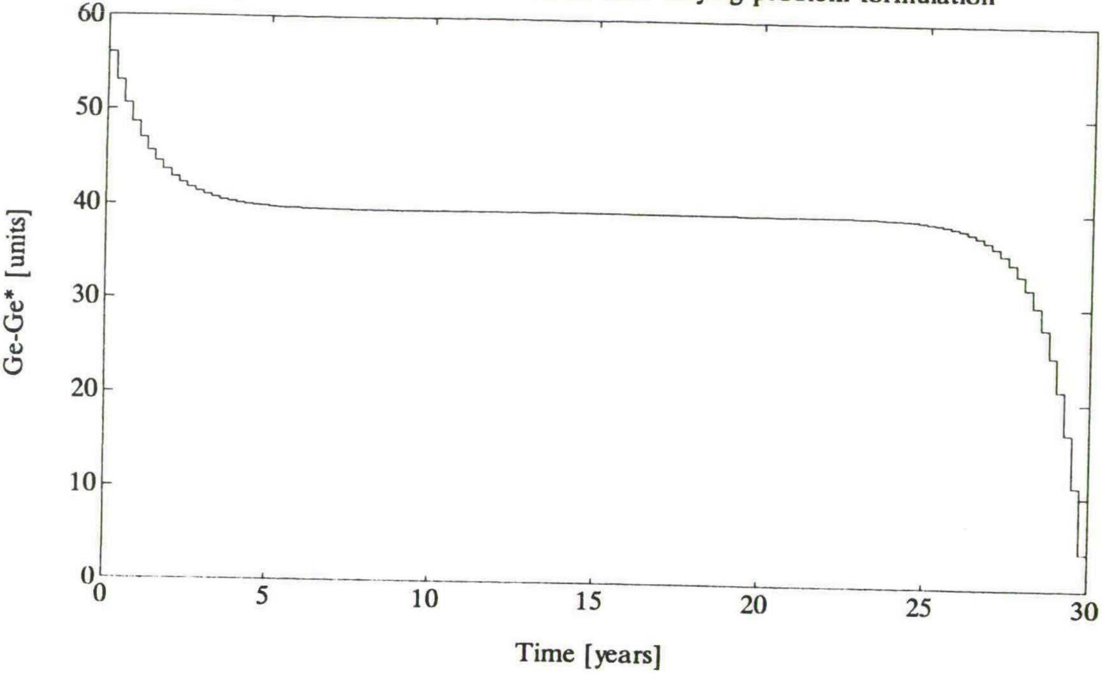


Figure 3b: Numerical solutions of time-varying problem formulation

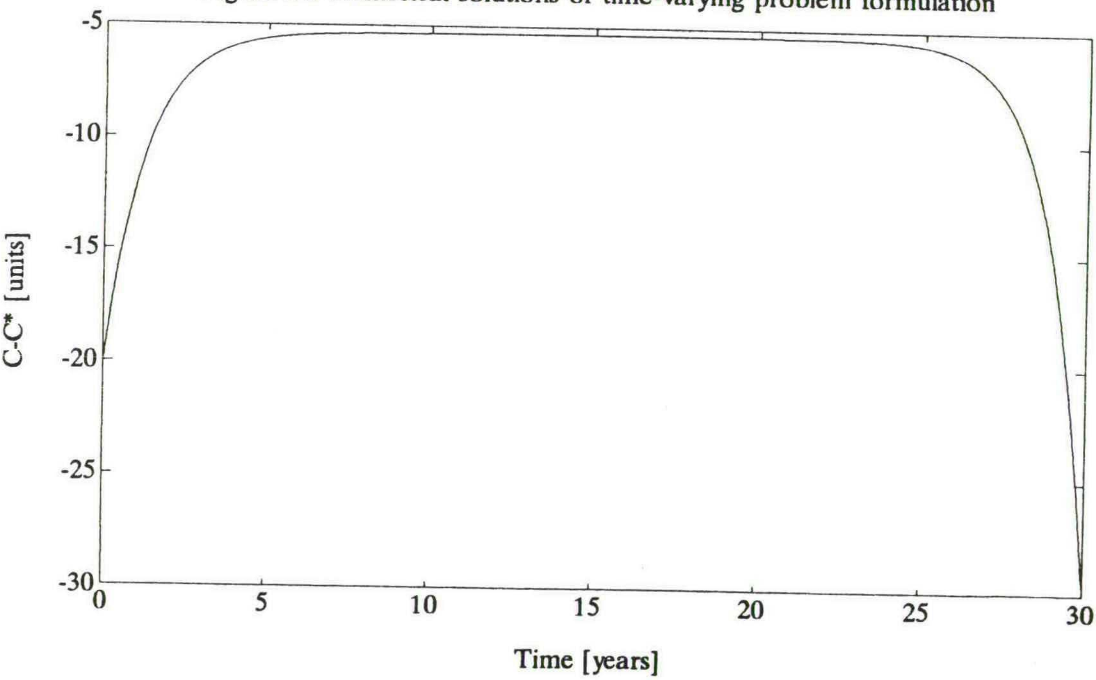


Figure 4a:Difference between solutions

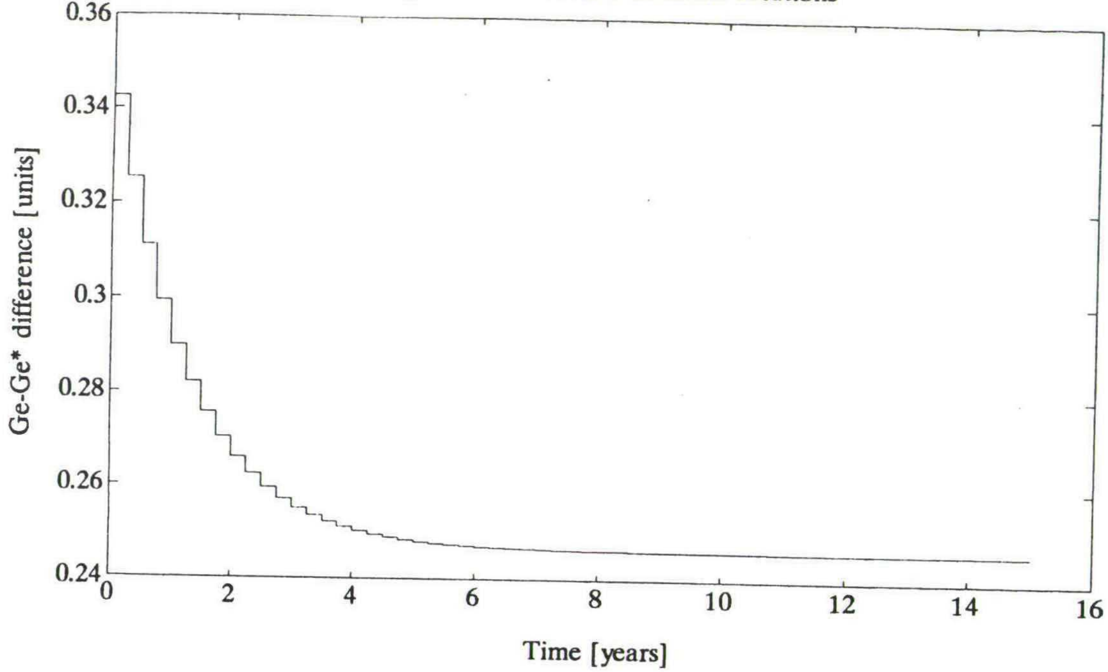
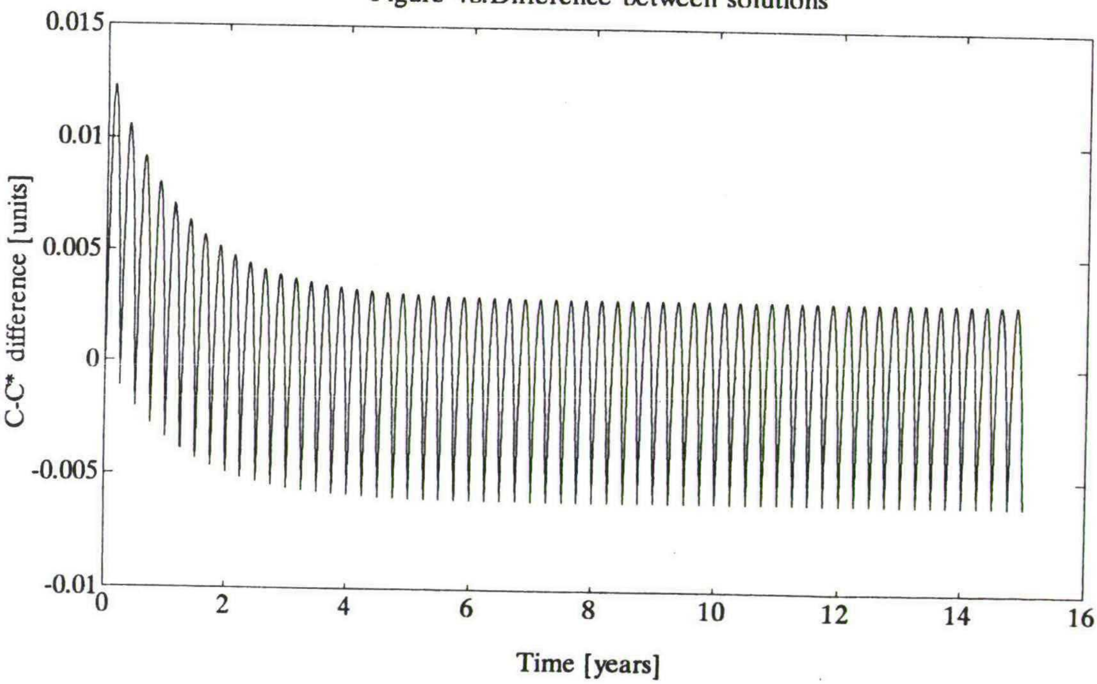


Figure 4b:Difference between solutions



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